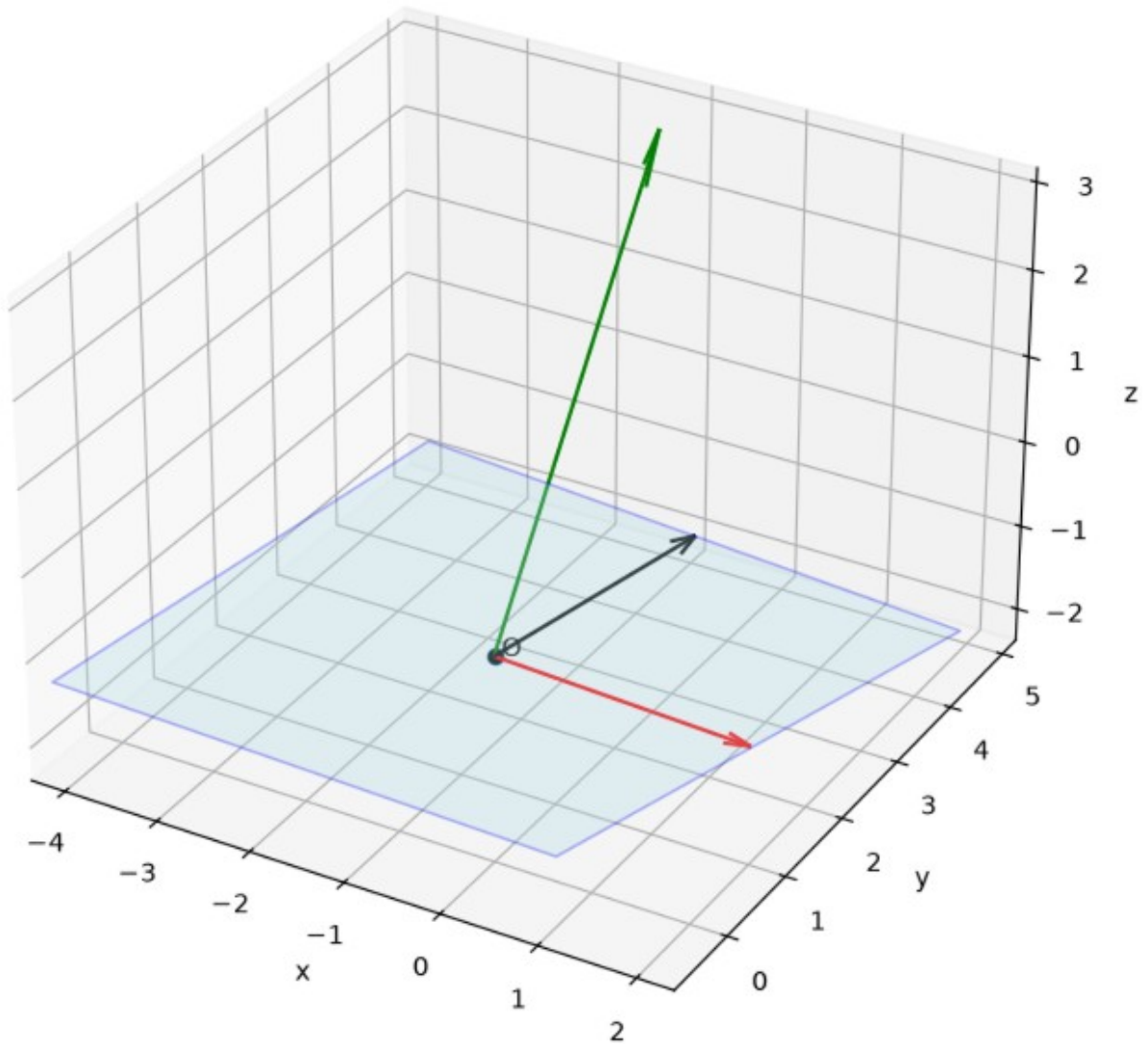


3 Space Geometry:

Lines, Planes, and Related Topics

*An Interactive Introduction to Vectors
and Three-Dimensional Geometry*

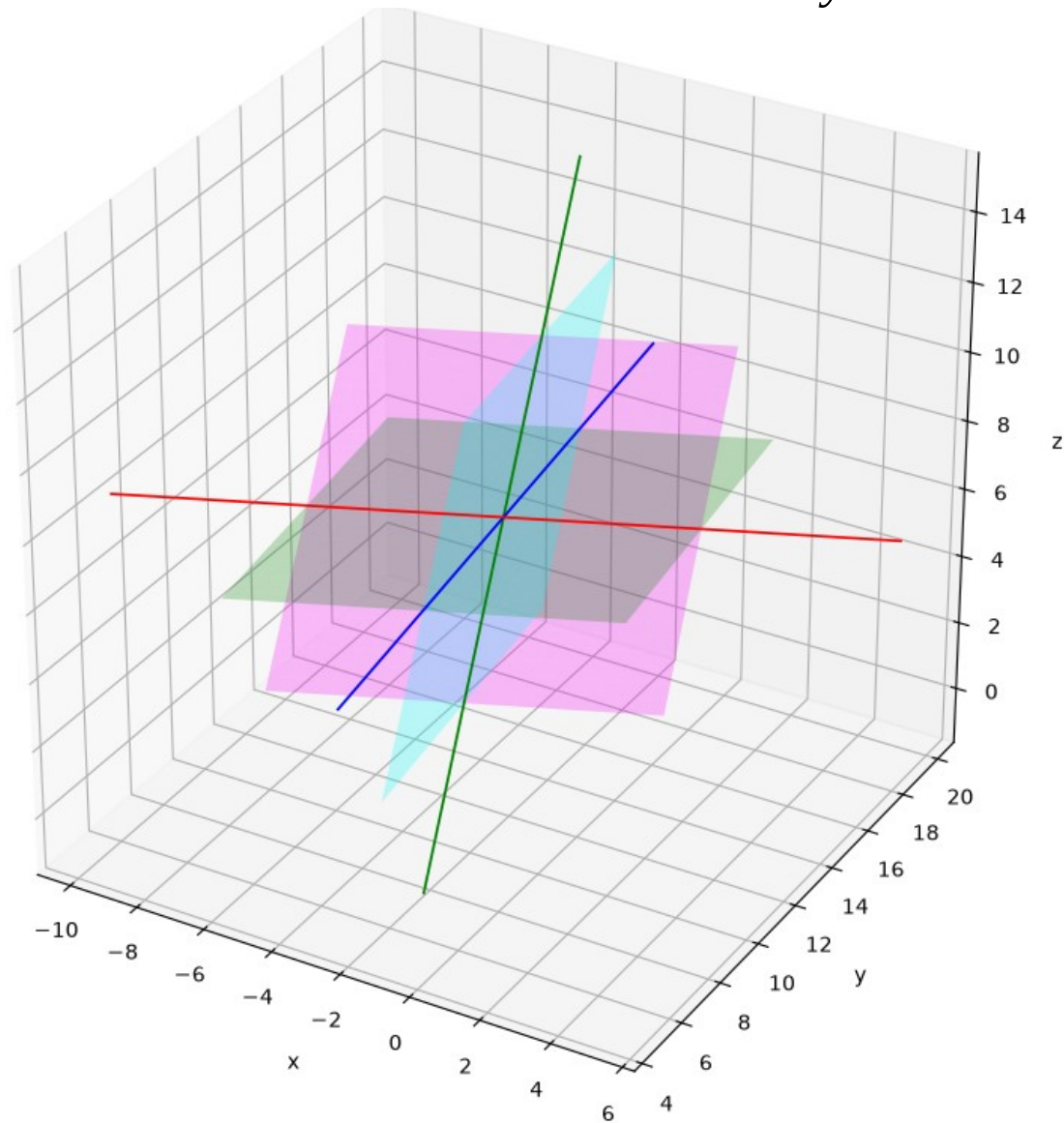


Companion Interactive Software Available

by Michael S. Runstein

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Preface:

This book is written to present lines in 3-space from a nontraditional perspective.

Lines are usually introduced in two-dimensional space and are often determined by two points. Their relationship is then examined through slope, commonly described as the rise over the run. This is an effective approach, and the three traditional forms of a line—slope-intercept form, point-slope form, and standard form—can all be derived from this viewpoint.

However, the traditional approach has limitations. While it works well in two-dimensional space, it becomes increasingly difficult to extend naturally to higher dimensions. As a result, many students encounter lines, planes, and higher-dimensional objects as separate topics rather than as parts of a unified geometric framework.

This book examines lines from a different perspective (or angle—pun intended). Rather than focusing on two points and slope, we will investigate lines using a point and a change from that point. This viewpoint not only applies in two-dimensional space, but extends naturally to three-dimensional space and beyond.

The first chapter provides a brief review of lines using the traditional methods commonly studied in algebra. The second chapter introduces an alternative viewpoint and shows how lines may be described using a point and a direction. Once this approach is understood, the study of planes and higher-dimensional geometric objects becomes much more natural.

The traditional mathematical curriculum typically progresses from elementary algebra through calculus before introducing points, lines, and planes in three-dimensional space, often in a vector calculus course. This book follows a different path. Beginning with familiar concepts, it gradually develops a geometric framework that extends naturally to n -space.

Along with this book, a Python program utilizing freely available software tools will be provided to aid visualization and understanding of the concepts being presented. The software is intended to serve as a companion to the text, allowing readers to explore examples, test ideas, and develop intuition through experimentation. Throughout the book, references will be made to corresponding software modules that reinforce and extend the material under discussion.

The goal is not merely to learn new formulas, but to develop a different way of thinking about geometry. By viewing lines, planes, and related objects through a common framework, many ideas from algebra, geometry, linear algebra, and calculus become more closely connected and easier to visualize.

I invite you to explore that journey.

Companion Software

A companion Python program has been written to accompany the book. The book covers mathematical topics beginning with lines in 2-space and culminating with vectors and planes in 3-space. While the book presents the underlying mathematics, the software allows the reader to experiment with different values and observe how those changes affect the graphs. Visualizing mathematics can be difficult when working with equations, particularly in three dimensions. The software is designed to help the reader see the math.

Every chapter, except Chapter 4, has a corresponding companion module. The reader can read the chapter first, use the corresponding module first, or examine them simultaneously. Examples are not provided. Experimentation is the recommended way to use the software. When calculations are required, the user can immediately see whether their answer is correct. Chapter 8 does not introduce any new mathematics. It allows the user to explore lines, planes, and vectors in 3-space and see how they are interrelated.

A section at the end of each chapter will explain how to use the companion module.

Additional Visualization Resources

Readers interested in additional visualization tools may also find freely available software such as Desmos, CalcPlot3D, and GeoGebra useful. These programs are not required for the material presented in this book, but they may provide alternative ways to explore and visualize mathematical concepts.

Chapter 1: Traditional 2-Space Lines

Let's look at lines in the traditional manner.

What is a line-segment? A line-segment consists of all points between two endpoints. Informally, we may think of a line-segment as a perfectly straight path connecting two points. Although this is not a formal mathematical definition, it provides a useful picture of a line-segment. Let the two points in the xy -plane be (x_0, y_0) and (x_1, y_1) .

A line extends the line-segment infinitely in both directions beyond the two endpoints.

Working in 2-space and the xy -plane, let us begin with the line and the two points previously described. There are three traditional forms used to describe lines: Standard Form, Point-Slope Form, and Slope-Intercept Form.

The Standard Form, $Ax + By = C$ or $Ax + By + C = 0$, can be used to represent any line in the plane. Vertical lines are easily written in standard form. For example, if x is constant, then $x = C/A$ represents a vertical line. Similarly, if y is constant, then $y = C/B$ represents a horizontal line. If the line is neither vertical nor horizontal, its equation in any of the three forms can be derived.

The Point-Slope and Slope-Intercept forms use one point and slope to define the line. Recall that slope, m , is defined as rise over run, or change in y divided by change in x : $m = (y_1 - y_0)/(x_1 - x_0)$. For example, if the two points are $(1, 2)$ and $(4, 8)$, the slope would be $m = (8 - 2)/(4 - 1) = 2$.

Both of these forms assume that x is not constant. If x is constant, then $(x_1 - x_0) = 0$ and the slope would be undefined since division by zero is not permitted. For this reason, vertical lines must be treated as a special case when using slope-based methods.

This limitation is often overlooked when first studying lines. Most lines can be represented using their slope, but vertical lines cannot. As we shall see in later chapters, there are alternative ways to describe lines that do not rely on slope, allowing vertical lines to be handled in exactly the same way as all other lines.

The first module of the companion Python program looks at the equations of a line in 2-space generated from two distinct points. The module presents the line and its equations in the traditional sense. As the book and companion program progress, we will look at lines in 2-space and 3-space from a different perspective.

Module 1

Module 1 is straightforward. Each Check button verifies whether the entered answer is correct. The Home button takes the user back to the Home screen of the program and the Exit button exits the program. These buttons are present in every module.

Chapter 2: Lines as a Point and Change in 2-Space

In chapter 1 we looked at lines from a traditional viewpoint.

In this chapter we will examine two-dimensional lines from a different perspective. Rather than viewing a line through its slope and equations, we will view a line as a point together with a constant change ratio (change ratios in later chapters) from that point. This viewpoint is common in Vector Calculus and extends to three dimensions and beyond. Subsequent chapters will build upon the idea that a line can be described by a point and constant change.

Throughout Pre-Calculus and introductory Calculus, lines are described as passing through a point and having a slope and associated equations. Unfortunately, the slope of a vertical line is undefined, which means that the point-slope and slope-intercept equations do not exist for vertical lines. We have generally accepted this fact without much thought. Nevertheless, vertical lines do exist and have equations; they simply do not have point-slope or slope-intercept equations. Since vertical lines exist but slope-based descriptions fail, is there a better way to describe all lines?

Every line passes through infinitely many points. As long as two points are known, the line can be defined. From those two points we can get the change in the y-coordinates and the x-coordinates from one point to the other. Equivalently, if we know one of the points and the change in each coordinate, we can determine the second point on the line. Traditionally, mathematics looks at a line and its equation from the viewpoints of two points and the slope. The slope is essential in analyzing the line and its properties. We propose looking at a line as one point and the change in coordinates from that point. We are not forgetting slope or the other parts of the line, such as the y-intercept. They simply are not necessary to describe a line in 2-space. Furthermore, in 3-space, slope and the y-intercept are not easily transferred. The point-and-change method contains as much information as the classical approach in 2-space and can be easily adapted to 3-space and higher dimensions.

Let's look at a few examples. Consider the point $(2, 3)$. Suppose that every time x increases by 4 units, y increases by 1 unit. Starting at $(2, 3)$, we can generate additional points by repeatedly applying these changes. $(2, 3)$, $(6, 4)$, $(10, 5)$, $(14, 6)$, and so on. Continuing this process we have an infinite number of points on the line. We could also go in the other direction $(-2, 2)$, $(-6, 1)$, $(-10, 0)$, and so on. But lines are continuous and do not consist of discrete points. So to fill in the intermediate points we need to consider fractional increases/decreases that have the same ratio as the original change ratio. Suppose that instead of increasing x by 4 units we increase it by 2 units. To remain on the line, y must increase by one-half unit. Similarly, if x increases by 1 unit, y must increase by one-fourth unit. The important thing is that the ratio of the changes remains constant. The ratio of the changes is the slope. However, for our purposes, it is enough to think of it simply as a constant relationship between the changes in the coordinates.

As a second example, consider the point $(2, 3)$ again. This time suppose that every time y increases by 4 units, x does not change. The points would be $(2, 3)$, $(2, 7)$, $(2, 11)$, and so on. Using a ratio to describe the pattern does not work in this example. However, the pattern is easily recognizable. The slope of this line is undefined since it is vertical. As a third example, suppose that every time x changes

by 4 units, y does not change. We would have (2, 3), (6,3), (10, 3), and so on. Although a ratio of change in y over change in x exists in this case, the line can still be described by a point and a change in coordinates. Ideally, we should be able to describe any line using a single formula.

Lets look at a formula that would work for the prior examples. Our starting point for the prior examples is the point (2, 3). We could say that for the first example the line would consist of the point (2, 3) plus multiples of (4, 1). More generally, $(x, y) = (2, 3) + t(4, 1)$, where t is any real number. Likewise, the second example would be $(x, y) = (2, 3) + t(0, 4)$ and the third example would be $(x, y) = (2, 3) + t(4, 0)$. t is used by convention, but any letter would work. Using delta, Δ , to indicate change, notation borrowed from Calculus, we could say that any line through the point (2, 3) could be written as $(x, y) = (2, 3) + t(\Delta x, \Delta y)$, where Δx and Δy are the changes in the x and y coordinates respectively.

A general definition of any line in the x-y plane through the point (x_0, y_0) is:

$$(x, y) = (x_0, y_0) + t(\Delta x, \Delta y).$$

It is no coincidence that the slope of any non-vertical line is given by $(\Delta y)/(\Delta x)$.

There are two other things that we need to consider. Let us call the point (x_0, y_0) the base point. First, what if we have a second point, (x_1, y_1) instead of than the change from the base point. That is easily handled by letting $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

The other question we need to address is how to find a point at a specific distance from the base point. We have already seen how to find points that are multiples, t, of the change, $(\Delta x, \Delta y)$. What if we want to find a point that is exactly a distance from the base point? We can scale the change so that it becomes a unit change, denoted by u, using

$$d = \sqrt{((\Delta x)^2 + (\Delta y)^2)}$$

and

$$(u(x), u(y)) = (\Delta x, \Delta y)/\sqrt{((\Delta x)^2 + (\Delta y)^2)} = (\Delta x/d, \Delta y/d)$$

Then the line through (x_0, y_0) can be defined by the equation:

$$(x, y) = (x_0, y_0) + t(u(x), u(y))$$

where t is a parameter that represents the unit distance traveled along the line from the base point, with positive and negative values corresponding to opposite directions.

Module 2

Module 2 has three options. The Point and Change Option allows the user to enter a base point and a change. The Two Point Option allows the user to enter two points, with the change determined by the difference between the points. The Point and Distance Option allows the user to enter a base point and a change and then determine the corresponding unit change. In each option, the user enters the equation of the line and checks whether it is correct. Only one option is active at a time.

Conclusion:

This book and the accompanying software are intended for the pre-calculus student. Chapter 1 and Module 1 started out by looking at lines in the manner that is traditionally taught. We then moved into looking at lines in a slightly different manner in both two and three dimensions. Terms like change and direction numbers were introduced. Chapter 5 and Module 5 introduced vectors and terms that are usually not covered until a vector calculus course. Chapters 6 and 7, together with Modules 6 and 7, explored additional topics usually not encountered in a pre-calculus course. Module 8 brings these topics together, connecting many of the concepts developed in the book and the visualizations in the software.

Many of the topics introduced in the latter chapters are covered in far greater detail in trigonometry, linear algebra, and vector calculus. Together, this book and the accompanying software are an introduction to subjects that have a wide array of applications, ranging from architecture and civil engineering to economics and, of course, mathematics. I hope you enjoy the software and find this book a useful introduction to topics that are usually not encountered before calculus.